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Magnetohydrodynamic Flow Past a Wedge with a Perpendicular Magnetic Field

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The magnetohydrodynamic two-dimensional steady supersonic flow past a wedge with attached straight shock waves is investigated. The fluid is assumed to be nonviscous and perfectly conducting. The wedge is nonconducting, nonmagnetic, and symmetric with respect to the flow. The magnetic field is applied perpendicular to the uniform flow. The general procedure to obtain the solution with two kinds of attached straight shock waves is studied for an arbitrary Mach number (>1) and an arbitrary strength of the magnetic field. The approximate solutions for the case of weak magnetic field and for the case of the small-angle shock waves are treated. The analysis shows that the hydrodynamic shock wave does not change its position and strength in the first order of the foregoing approximation. However, when the weak magnetic field is applied, the new magnetohydrodynamic shock wave appears near the surface of the wedge, and it leaves the surface as the magnetic field increases; for a further increase of the magnetic field, it meets the hydrodynamic shock wave, and then the solution with attached shock wave disappears. For a still stronger magnetic field, the magnetohydrodynamic shock wave appears in front of the hydrodynamic shock wave.

1. Introduction

THE magnetohydrodynamic two-dimensional steady supersonic flow past a wedge with attached straight shock waves is investigated. The fluid is nonviscous and perfectly conducting. The wedge is insulated and nonmagnetic; it is placed symmetric to the flow. The magnetic field is applied perpendicular to the uniform flow. Kogan[1] and Chu[2] treated this problem including unsymmetric flows with the linearized theory. They obtained solutions with two kinds of attached waves. The nonlinear problem is treated here for an arbitrary Mach number (>1) and for arbitrary magnetic field strength.

2. Helfer's Treatment of a Shock Wave

The relations between the physical quantities across the shock waves in a perfectly conducting, compressive, nonviscous fluid are determined. Following de Hoffman and Teller,[3] the flow velocity is made parallel to the magnetic field by using a coordinate system moving parallel to the shock front.

The y axis is chosen parallel to the shock front and is pointed toward the apex of the wedge, and the x axis is perpendicular to the y axis and pointed toward the wedge. Then the coordinate system on the lower side of the wedge is a right-hand one (Fig. 1). Physical quantities in front of and behind the shock wave are discriminated by the suffixes 1 and 2. The magnetic field is denoted by $\mathbf{B}_i$ ($i = 1, 2$), the angle made by the magnetic field to the normal of the shock front by θ_i, the fluid velocity by $\mathbf{v}_i$, the density by ρ_i, the pressure by p_i, and the permeability by μ, and it is assumed that μ is constant (Fig. 2) (mks units are used).

Choosing the coordinate system in which $\mathbf{B}_i$ and $\mathbf{v}_i$ are parallel,

$$B_{iy}/B_{ix} = v_{iy}/v_{ix} \qquad i = 1,2 \quad (2.1)$$

The continuity equations, momentum equations, and energy equation are given by

$$\rho_1 v_{1x} = \rho_2 v_{2x} = k \quad (2.2)$$

$$B_{1x} = B_{2x} \quad (2.3)$$

$$k(v_{2x} - v_{1x}) = (p_1 + B_{1y}^2/2\mu) - (p_2 + B_{2y}^2/2\mu) \quad (2.4)$$

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$$k(v_{2y} - v_{1y}) = (1/\mu)(B_{2x}B_{2y} - B_{1x}B_{1y}) \tag{2.5}$$

$$\tfrac{1}{2}v_1^2 + [\gamma/(\gamma - 1)]p_1/\rho_1 = \tfrac{1}{2}v_2^2 + [\gamma/(\gamma - 1)]p_2/\rho_2 \tag{2.6}$$

where $v_i = |\mathbf{v}_i|$, $B_i = |\mathbf{B}_i|$, and γ is the ratio of specific heats. To derive (2.6), the internal energy is put equal to $(\gamma - 1)^{-1}p\rho^{-1}$.

The calculations and the results are very much simplified if the specific heat is chosen to be $\frac{5}{3}$ as shown by Helfer.[4] To make the calculation for the case of other arbitrary values of γ as simply as the case of $\gamma = \frac{5}{3}$, the symbolic integers n^* are used, which are defined by $1^* = 2(2 - \gamma)/(\gamma - 1)$ and $n^* + 1 = (n + 1)^*$, where n is a positive integer. Thus,

$$1^* = 2\frac{2 - \gamma}{\gamma - 1} \qquad 2^* = \frac{3 - \gamma}{\gamma - 1} \qquad 3^* = \frac{2}{\gamma - 1}$$
$$4^* = \frac{\gamma + 1}{\gamma - 1} \qquad 5^* = \frac{2\gamma}{\gamma - 1} \qquad \text{etc.} \tag{2.7}$$

When γ takes the value $\frac{5}{3}$, 1^* takes the value one, so that n^* takes the value n. Namely, the right-hand sides of (2.7) are expressed symbolically using the numerical values for the case of $\gamma = \frac{5}{3}$. Since $\gamma \equiv 5^*/3^*$, any formula containing γ can be expressed by using symbolic integers. The symbolic integers n^* satisfy the addition rule $n^* + m = (n + m)^*$, but care should be taken that they do not satisfy the multiplication rule, namely, $n \cdot m^* \neq (nm)^*$, because

$$n \cdot m^* - (nm)^* = [n \cdot 1^* + nm - n] - [1^* + nm - 1] = (n - 1)(1^* - 1) \neq 0$$

except $n = 1$ or $\gamma = \frac{5}{3}$.

It is convenient to treat the shock relations (2.4–2.6) following Helfer.[4] The necessary parts used for the present problem are summarized below. Some notation is changed from that of Helfer's. Nondimensional variables are defined using the quantities on the same side of the shock front ($i = 1$ or 2):

$$B_{iy}/B_{ix} = v_{iy}/v_{ix} = \tan\theta_i = s_i \tag{2.8}$$

$$m_i = v_i/\alpha_i \qquad \alpha_i = B_i/(\mu\rho_i)^{1/2} \tag{2.9}$$

$$Q_i = B_i^2/2\mu p_i = (5^*/2 \cdot 3^*)\alpha_i^2 c_i^2 \tag{2.10}$$

where α_i is the Alfvén wave velocity, and $c_i = (5^*p_i/3^*\rho_i)^{1/2}$ represents the sound velocity when the magnetic field is not applied. The quantities s_1 and Q_1 are used frequently, so that the suffix is omitted for these two.

The following quantities are defined using the ratio of the quantities in front of and behind the shock wave:

$$\rho_2/\rho_1 = v_{1x}/v_{2x} = m_1^2/m_2^2 = \eta(>1) \tag{2.11}$$

$$p_2/p_1 = 1 + X \tag{2.12}$$

$$s_2/s = x \tag{2.13}$$

$$B_2^2/B_1^2 = T^2 \tag{2.14}$$

The relations (2.11) are derived using (2.2, 2.3, and 2.8). The increase of entropy requires the condition $\eta > 1$.[5] Helfer[4] showed that $X > 0$ when $\eta > 1$.

Using the parameters η and x, Helfer's parameter is defined as

$$A = (\eta - 1)/(\eta - x) \tag{2.15}$$

Then one has

$$\eta = (1 - xA)/(1 - A) \tag{2.16}$$

The following three relations are easily verified:

$$v_{ix}^2 = 2m_i^2 Q_i p_i/\rho_i(1 + s_i^2) \tag{2.17}$$

$$Q_2/Q = T^2/(1 + X) \tag{2.18}$$

$$T^2 = (1 + x^2s^2)/(1 + s^2) \tag{2.19}$$

where (2.3) was used to derive (2.19).

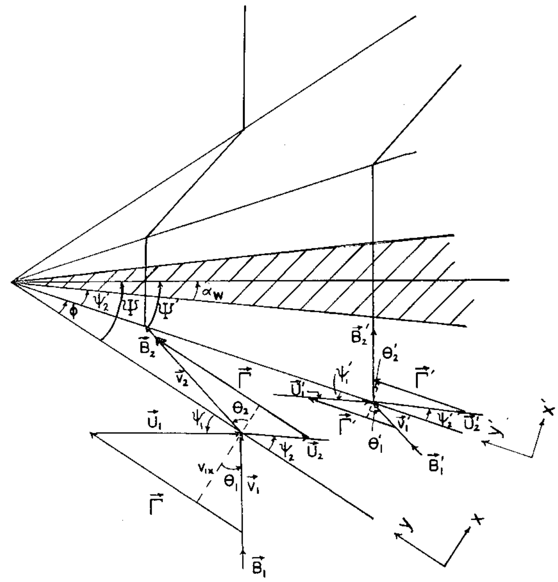

Fig. 1 Flow past a wedge.

Using these relations and (2.1–2.3), the shock relations of (2.4–2.6) are made nondimensional, and they are expressed, respectively, as

$$Q = (1 + s^2)X/[2m_1^2(1 - \eta^{-1}) + s^2(1 - x^2)] \tag{2.20}$$

$$m_1^2 = 1 - xA \tag{2.21}$$

$$Qm_1^2 = (5^*/2)[\eta^{-1}(1 + X) - 1]/[1 - (T^2/\eta^2)] \tag{2.22}$$

Equation (2.20) is transformed to

$$X = [(1 - x)/(1 + s^2)][2A + (x + 1)s^2]Q \tag{2.23}$$

by using (2.16) and (2.21). This value of X is inserted into (2.22), and (2.16, 2.21, and 2.19) are used to get

$$Q = 5^*(1 + s^2)A/\{2(x - 4^*)A^2 + [2 \cdot 3^* - (1^*x + 5^*)s^2]A + 3^*(x + 1)s^2\} \tag{2.24}$$

The quantities in front of and behind the shock wave are determined when Q, s, and x are given. In this case, A is given by Eq. (2.24):

$$A_\pm = [E/4(x - 4^*)Q]\{1 \pm [1 - 8 \cdot 3^*(x - 4^*)(x + 1)s^2Q^2/E^2]^{1/2}\} \tag{2.25}$$

where

$$E = 5^*(1 + s^2) - 2 \cdot 3^*Q + (1^*x + 5^*)Qs^2$$

The values $X_\pm$ are determined by (2.23) and $A_\pm$. Then, using (2.18, 2.19, and 2.16), one finds

$$Q_{2\pm} = [Q/(1 + X_\pm)](1 + x^2s^2)/(1 + s^2) \tag{2.26}$$

$$\eta_\pm = (1 - xA_\pm)/(1 - A_\pm) \tag{2.27}$$

The other nondimensional quantities can be determined easily.

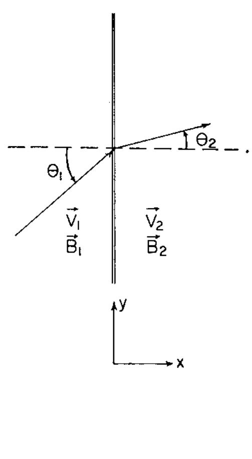

Fig. 2 Quantities in front of and behind the shock wave.

All quantities are determined uniquely when Q, s, and x are given and one of the values A_+ or A_- is chosen.

If other sets of these quantities were given, a third- or higher-degree equation would have to be solved. This is the important merit of Helfer's parameter A. The parameter A is expressed by using (2.11, 2.16, 2.21, and 2.9)as

$$A = 1 - m_2^2 = (\alpha_2^2 - v_2^2)/\alpha_2^2 \tag{2.28}$$

and it represents a measure of the inequality between the kinetic and the magnetic energies per unit volume.[4]

3. Limiting Cases

1. Weak Magnetic Field

The case of $Q \ll 1$ is examined. This has been treated already by Helfer. In this case, using (2.25), $|A_+| \gg 1$ or $|A_-| \ll 1$. Expanding $1/A_+$ by Q, one has

$$1/A_+ = -A_+^{(1)}Q(1 + A_+^{(2)}Q + \ldots) \tag{3.1}$$

where

$$A_+^{(1)} = (2/5^*)(4^* - x)/(1 + s^2) \tag{3.2}$$

$$A_+^{(2)} = [2\cdot 3^* - (1^*x + 5^*)s^2]/5^*(1 + s^2) \tag{3.3}$$

Using (3.1) and (2.27), one obtains

$$\eta_+ = x - (2/5^*)[(4^* - x)(x - 1)/(1 + s^2)]Q \tag{3.4}$$

where the terms up to the order of Q have been retained. Also one has, using (2.25),

$$A_- = (3^*/5^*)[(x + 1)s^2/(1 + s^2)]Q \tag{3.5}$$

Hence, by (2.27),

$$\eta_- = 1 + (1 - x)A_- \tag{3.6}$$

The shock wave corresponding to $A = A_+$ reduces to the ordinary hydrodynamic shock wave in the case $Q = 0$, so that this is the ordinary hydrodynamic shock wave modified by the magnetic field. This is called the "OHD shock wave." In the case $A = A_-$, the pressure change and density change are $0(Qs^2)$ by (3.5, 3.6, and 2.23), so that they are small when Qs^2 is small. However, x remains finite, so that the direction and the strength of the magnetic field change. This shock wave is well known as the characteristic shock wave of magnetohydrodynamics, and it is called the "MHD shock wave."

2. Small-Angle Shock Waves

Assume that $s^2 \ll 1$, and seek solutions up to the lowest-order terms in s^2. The following equality and inequality relations are satisfied when the next-lowest-order terms of s^2 are neglected. Assume that

$$5^* - 2\cdot 3^*Q = 0(s^0) \tag{3.7}$$

$$|x| \leq 0(s^0) \tag{3.8}$$

$$4^* - x = 0(s^0) \tag{3.8a}$$

Equation (3.7) means that the value of Q which satisfies the relation $5^* - 2\cdot 3^*Q = 0(s^2)$ is avoided. Then one obtains

$$8\cdot 3^*\,(x - 4^*)(x + 1)s^2Q^2/E^2 = 0(s^2)$$

in (2.25), so that

$$A_+ = -(5^* - 2\cdot 3^*Q)/2(4^* - x)Q$$

$$\tag{3.9}$$

$$A_- = [3^*(x + 1)/(5^* - 2\cdot 3^*Q)]Qs^2 \tag{3.10}$$

In the case $A = A_+$, one has, using (2.23),

$$X_+ = 2(1 - x)A_+Q = (5^* - 2\cdot 3^*Q)(x - 1)/(4^* - x) \tag{3.11}$$

Solving this equation for x, one has

$$x = (Q_{c+} - Q)/(Q_{b+} - Q) \tag{3.12}$$

where

$$\begin{aligned} Q_{b+} &= (1/2\cdot 3^*)(5^* + X_+) \\ Q_{c+} &= (1/2\cdot 3^*)(5^* + 4^*X_+) \end{aligned} \tag{3.13}$$

By (3.11) and (3.12),

$$m_{2+}^2 = 1 - A_+ = Q_{b+}/Q \tag{3.14}$$

$$m_{1+}^2 = 1 - xA_+ = Q_{c+}/Q \tag{3.15}$$

Using (2.27), one finds

$$\eta_+ = Q_{c+}/Q_{b+} \tag{3.16}$$

In the case of $A = A_-$, one has, using (3.10) and (2.27) and retaining here the next order in s^2,

$$\eta_- = 1 + (1 - x^2)[3^*/(5^* - 2\cdot 3^*Q)]Qs^2 \tag{3.17}$$

From (2.23),

$$X_- = (1 - x^2)[5^*/(5^* - 2\cdot 3^*Q)]Qs^2 \tag{3.18}$$

It is clear from (3.16) or (3.17) that the shock wave corresponding to A_+ or A_- is the OHD or the MHD shock wave, respectively.

4. General Procedure for Flow Past a Wedge

Now the magnetohydrodynamic two-dimensional steady supersonic flow past a wedge with attached straight shock waves is investigated. The wedge is assumed to be nonconducting, nonmagnetic, and placed symmetric with respect to the flow. The magnetic field is applied perpendicular to the uniform flow. The fundamental shock relations (2.1–2.6) are independent of the sign of the magnetic field, so that the flow field and the magnetic field, except its direction, are symmetric, and magnetic field in the wedge is perpendicular to the axis of the wedge. As is well known, an electric-current sheet at the surface of the wedge does not exist in this configuration, because, if it did, the fluid would receive an infinitely large acceleration. Hence, the magnetic field passes the surface of the wedge without bending. The directions of the magnetic-field lines and the flow velocity change at the shock fronts only.

Since the magnetic field is perpendicular to the axis in the uniform flow and in the wedge, the field lines must be bent by at least two shock waves when they pass from the uniform flow to the wedge. This is illustrated as follows. When the upper and lower magnetic-field lines are bent by the upper and lower hydrodynamic shock waves, respectively, and enter into the wedge, they make a kink on the axis. The kink is stretched by the magnetic tension, and this causes a new shock wave of electromagnetic origin; the two kinds of shock waves affect each other and make a steady flow pattern. The steady solution is sought with two kinds of attached shock waves in which the flow at the surface of the wedge is parallel to the surface, and the magnetic field is perpendicular to the axis in the body. The preceding shock wave is called the fast one and the other, the slow one.

The physical quantities in front of and behind the shock wave are defined as in Sec. 2, and a prime is put on the quantities related to the slow shock wave. The uniform velocity measured in the coordinate system fixed to the wedge is denoted by $\mathbf{U}_i$, and the angle between $\mathbf{U}_i$ and the shock front is denoted by ψ_i (Fig. 1). If the moving coordinate system with the velocity $-\mathbf{\Gamma}(\Gamma_x = U_1 \sec\theta_1, \Gamma_y = 0)$ is used, the flow velocity is transformed to $\mathbf{v}_1 = \mathbf{U}_1 + \mathbf{\Gamma}$ and is parallel to the magnetic field, since $\mathbf{U}_1$ is perpendicular to $\mathbf{B}_1$. In this coordinate system, the flow velocity $\mathbf{v}_2$ behind the fast shock wave is also parallel to the magnetic field $\mathbf{B}_2$. When the flow velocity $\mathbf{v}_2$ is measured in the coordinate system fixed to the wedge, it is transformed to the velocity $\mathbf{U}_2$ given by the relation $\mathbf{U}_2 =$

$\mathbf{v}_2 - \mathbf{\Gamma}$. The physical quantities that are independent of the angle in front of the slow shock wave are equal to the corresponding ones behind the fast shock wave. However, the velocity of the moving coordinate system of the slow shock wave should be parallel to the y' axis so that $\mathbf{\Gamma} \neq \mathbf{\Gamma}'$ and $\mathbf{v}' = \mathbf{U}' + \mathbf{\Gamma}' \neq \mathbf{v}_2'$ (Fig. 3).

1) To treat the whole flow, it is convenient to give the quantities Q, s, and x at the fast shock front as explained in Sec. 2. In this case, the Mach number M_1 in the uniform flow is subsequently found from

$$M_1 = (v_1/s)/c_1 = (1/s)[(2\cdot 3^*/5^*)Qm_1{}^2]^{1/2} \tag{4.1}$$

and ψ_1 is given by

$$\tan\psi_1 = s \tag{4.2}$$

One determines ψ_2 from s, x, η, and ψ_1. One obtains two geometrical relations involving Γ from Fig. 1:

$$\tan\psi_2 = v_{2x}/(\Gamma - v_{2y})$$

and

$$\Gamma = (v_{1x}/\tan\psi_1) + v_{1x}\tan\theta_1 \tag{4.3}$$

where $\Gamma = |\mathbf{\Gamma}|$. From these equations and (3.4, 2.11, and 2.13), one has

$$\tan\psi_2/\tan\psi_1 = [\eta(1 + s\tan\psi_1) - xs\tan\psi_1]^{-1} \tag{4.4}$$

The foregoing equation is satisfied for the slow shock wave if a prime is put on every quantity, but $\tan\psi_1' \neq s'$ because $\mathbf{U}_1'$ is no longer perpendicular to $\mathbf{B}_1'$.

2) The relations between the angles involving both the fast and slow shock waves are given by (see Fig. 1)

$$\theta_1' = \theta_2 - \phi \tag{4.5}$$

$$\theta_2' = \theta_1 - \phi \tag{4.6}$$

$$\psi_1' = \psi_2 - \phi \tag{4.7}$$

where ϕ is the angle between the two shock waves. To derive (4.6), the condition that the magnetic field lines are perpendicular to the axis at the wedge was used. From (4.5) and (4.6) one has, using (2.8) and (2.13),

$$s' = (xs - \tan\phi)/(1 + xs\tan\phi) \tag{4.8}$$

$$s_2' = (s - \tan\phi)/(1 + s\tan\phi) \tag{4.9}$$

$$x' = (1 + xs\tan\phi)(s - \tan\phi)/(1 + s\tan\phi)(xs - \tan\phi) \tag{4.10}$$

If ϕ is given, then s' and x' are determined by the foregoing equations. Moreover, $Q' = Q_2$. Hence, using Q', s', and x', one could determine $A_\pm'$, and then for each A_+' or A_-' all quantities related to the slow shock wave are determined uniquely as in the fast shock.

3) To determine ϕ or ψ_1', $\mathbf{U}_2 = \mathbf{U}_1'$ is used. Namely, using a geometrical relation in Fig. 3, one obtains

$$v_{1x}'/v_{2x} = \sin\psi_1'/\sin\psi_2 \tag{4.11}$$

On the other hand, from (2.17) and (2.11),

$$v_{1x}{}'^2/v_{2x}{}^2 = (v_{1x}{}^2/v_{2x}{}^2)(v_{1x}{}'^2/v_{1x}{}^2)$$

$$= \eta^2 m_1'^2 Q' p_1' \rho_1 (1 + s^2)/m_1{}^2 Q p_1 \rho_1' (1 + s'^2)$$

Since $p_1' = p_2$, $\rho_1' = \rho_2$, and $Q' = Q_2$, using (2.27, 2.21, 2.18 2.12, and 2.11), one has

$$v_{1x}{}'^2/v_{2x}{}^2 = [(1 - x'A')/(1 - A)]T^2[(1 + s^2)/(1 + s'^2)] \tag{4.12}$$

Hence, from (2.19) and (4.11), one has

$$(1 + s'^2)\sin^2\psi_1'/(1 - x'A') = (1 + x^2s^2)\sin^2\psi_2/(1 - A) \tag{4.13}$$

When Q, s, and x are given, the angle ϕ is determined by this equation, as it is contained only in the left-hand side in a complex form of elementary functions.

4) One determines ψ_2' by applying (4.4) to the slow shock wave. Then the half angle α_w of the apex of the wedge is determined by

$$\alpha_w = \psi_1 - \phi - \psi_2' = \psi_1 - \psi_2 + \psi_1' - \psi_2' \tag{4.14}$$

Thus, α_w is determined for the given set of values of Q, s, and x, and the whole flow field has been solved.

If the angles between the fast or slow shock waves and the axis of the wedge are denoted by Ψ and Ψ', then

$$\Psi = \psi_1 = \tan^{-1}s \tag{4.15}$$

$$\psi' = \alpha_w + \psi_2' = \Psi - \phi \tag{4.16}$$

5) By combining $A_\pm$ and $A_\pm'$ at the fast and slow shock waves, one can get four possible types of flow solution. The flow with A_+ and A_- is called the OHD-MHD type, and MHD-OHD, OHD-OHD, and MHD-MHD types of flow are defined in the same way. In order that they be physically possible solutions, it is necessary that $\phi > 0$ and $\alpha_w > 0$, in addition to $X > 0$ and $X' > 0$.

6) In most of the actual problems, the Mach number M_1, the ratio of magnetic pressure to the static pressure Q, and the half angle α_w of the apex of the wedge are given. To solve this problem, s is chosen arbitrarily, and $m_1{}^2$ is determined from (4.1):

$$m_1{}^2 = (5^*/2\cdot 3^*)M_1{}^2s^2/Q \tag{4.17}$$

Next, x is eliminated from (2.21) and (2.24), and a third-degree equation is obtained for A in which the coefficients are functions of M_1, Q, and x. Using (2.21), one then determines x. One can, if desired, determine + or − of A in the form of (2.25) by inserting Q, s, x, and A into (2.25). Following procedures 1–4, one can determine α_w as a function of s. One selects s so that α_w is equal to the given value.

5. Weak Magnetic Field

To treat the flow past a wedge analytically, first one assumes $Q \ll 1$ and calculates the lowest-order terms of the physical quantities involving Q. The shock may be strong. The approximate formulas derived in Sec. 3.1 are used. It can be expected that the fast shock wave may be the OHD shock wave because of the small Alfvén speed, but this will be proved at the start. At the fast shock wave, if M_1 is kept constant and $Q \to 0$, one has $m_1{}^2 \to \infty$ by (4.1). Then $xA \to -\infty$ by (2.21). Hence, $|x| \to \infty$ or $|A| \to \infty$. From (2.25), one has a finite $A_\pm$ when $|x| \to \infty$. Hence, by (2.23), $X < 0$.

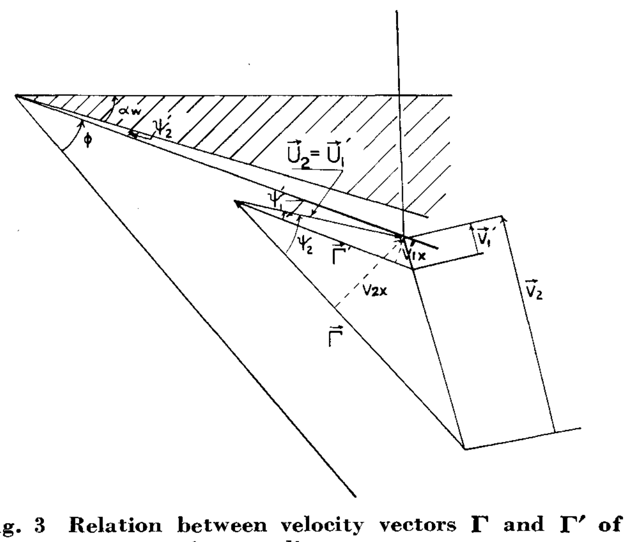

Fig. 3 Relation between velocity vectors $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ of moving coordinate systems.

This is physically improbable. The remaining possibility is $|A| \to \infty$; this is an OHD shock wave corresponding to (3.1). It is known that the fast shock wave is an OHD shock wave. Notice that it is an OHD shock wave if $m_1^2 \to \infty$.

Next it is shown that the slow shock wave is an MHD shock wave. In the case of $Q \ll 1$, one may use the first term of (3.1) only and insert it into (2.27). Then, using $\eta > 1$, one has $x > 1$. One sees that $Q \ll 1$, and $x > 1$ means that it corresponds to the OHD shock wave. Transforming (4.10), one has

$$[(1 - x')/(x - 1)](xs - \tan\phi) = s(1 + \tan^2\phi)/(1 + s\tan\phi) > 0 \quad (5.1)$$

Noting that $s > \phi$, one has $x' < 1$. Using (2.26), one has

$$Q' = Q_2 \leqq 0(Q) \quad (5.2)$$

because $T^2 = 0(Q^0)$ by (2.19). It is clear that $Q' \ll 1$, and $x' < 1$ means that the slow shock wave is an MHD shock wave.

The slow MHD shock wave is now examined. Applying (3.5) to this wave, one has

$$A' = A_-' = (3^*/5^*)(x' + 1)s'^2(1 + s'^2)^{-1}Q' \quad (3.5')$$

where a prime attached to the number of the formula means that the formula is applied to the slow shock wave. Using (4.13, 3.1, 2.19, and 3.5′), one has

$$\sin^2\psi_1'/\cos^2(\theta_2 - \psi_2 + \psi_1') = (2/5^*)(4^* - x)T^2(\sin^2\psi_2)Q$$

where one uses the relation $1 + s'^2 = \sec^2(\theta_2 - \psi_2 + \psi_2')$, which is derived from (4.5) and (4.7). Hence, $\psi_1' = 0(Q)$, and then $\theta_2 - \psi_2 = 0(Q^0)$. Using this estimation, one has

$$\psi_1' = (2/5^*)^{1/2}(4^* - x)^{1/2}T\cos(\theta_2 - \psi_2)(\sin\psi_2)Q^{1/2} \quad (5.3)$$

Applying (4.4) to the slow wave and using (3.5′) and (3.6′),

$$\tan\psi_2'/\tan\psi_1' = [1 + s'(1 - x')\tan\psi_1']^{-1}$$

Using $\psi_1' = 0(Q)$, one has

$$\psi_1' - \psi_2' = s'(1 - x')\psi_1'^2 + 0(Q^{3/2}) \quad (5.4)$$

Hence, $\psi_1' \propto Q^{1/2}$, $\psi_2' \propto Q^{1/2}$, and $\psi_1' - \psi_2' \propto Q$. Since $\psi_2' = \Psi' - \alpha_w$, where ψ_2' is the angle between the wall and MHD shock wave, the slow shock wave moves away from the surface of the wedge at an angle proportional to $Q^{1/2}$.

Since $A_-' = 0(Q')$, one has $m_1^2 = 1 + 0(Q')$ or $\alpha_1' = v_1'[1 + 0(Q')]$ from (2.21) and (2.9). Referring to Fig. 1, it is noted that $\mathbf{U}_1' - \mathbf{v}_1'$ is equal to $\mathbf{\Gamma}'$ and parallel to the shock wave. Hence, the MHD shock wave may be considered to develop from the Alfvén wave, which propagates along $\mathbf{B}_1'$ with the Alfvén velocity α_1' and leaves from the apex of the wedge in the flow of velocity $\mathbf{U}_1'$. The propagation velocity of this MHD shock wave, v_{1x}', is given by $\alpha_1' \cos\theta_1'$, which corresponds to the Friedrichs diagram when $Q \ll 1$[6]. However, the linear theory in which one puts $\mathbf{B}_2' = \mathbf{B}_1' + \delta\mathbf{B}'$ and $\delta B' \ll B_1'$ fails to give the slow wave when α_w is finite and $Q \to 0^2$. This is because $\mathbf{B}_2' - \mathbf{B}_1'$ is the same order as B_1', since $x - 1$, and so $1 - x'$ is nonvanishing and finite if α_w is finite, even in the limit of $Q \to 0$.

It is interesting to note the reason why there can be a finite deflection of the magnetic-field line, without any remarkable effect on the flow field, at the slow wave in the limit $Q \to 0$. It was noted that $m_1'^2 = v_1'^2/\alpha_1'^2 \to \infty$ corresponds to an OHD shock wave only, and in this case $\alpha_1' \to 0$. However, in the limit $Q \to 0$, $\mathbf{U}_1'$ is almost parallel to the surface of the wedge, so that, referring to Fig. 3, in which the slow wave makes a small angle to the surface, the component of $\mathbf{U}_1'$, parallel to the magnetic field $\mathbf{v}_1'$, is small, and $v_1'^2/\alpha_1'^2$ can take on a value near to one. Hence the slow wave can be an MHD shock wave, and the magnetic-field line can be deflected through a finite angle.

6. Small-Angle Shock Waves

1. The Case $s^2 \ll 1$

This is the interesting case where the Mach number is large and the wedge angle is small. As in Sec. 3.2, one calculates the lowest-order terms in s^2 of the solutions, and the equality and inequality relations are satisfied when the next-lowest-order terms in s^2 are neglected. The formulas in Sec. 3.2 are used, assuming the conditions (3.7, 3.8, and 3.8a).

One obtains $T^2 = 1$ from (2.19). Using this relation and (2.26), one has

$$Q_2 = Q_1/(1 + X) \quad (6.1)$$

From (4.4) and (4.2), putting $\tan\psi_2$ as ψ_2, one finds

$$\psi_2/s = 1/\eta \quad (6.2)$$

From (4.8) and (3.8), since $\phi < s$,

$$s' = xs - \phi = 0(s) \quad (6.3)$$

From (4.13, 6.2, 6.3, and 3.8), one has

$$\psi_1'/s = (1/\eta)[(1 - x'A')/(1 - A)]^{1/2} \quad (6.4)$$

Hence, from (4.7, 6.2, and 6.4),

$$\phi/s = \psi_2/s - \psi_1'/s = (1/\eta)[1 - (1 - x'A')^{1/2}/(1 - A)^{1/2}] \quad (6.5)$$

Using the approximation $\tan\phi = \phi$, (4.7) and (6.2), one has, from (4.10),

$$x' = (1 - \phi/s)/(x - \phi/s) = [1 - (1/\eta) + (\psi_1'/s)]/[x - (1/\eta) + (\psi_1'/s)] \quad (6.6)$$

Following the general procedure stated in Sec. 4, Q, x, and s are given at first, and A, η, and A' are determined. Then, combining (6.4) and (6.6), one gets x' and ψ_1'/s. Hence, ϕ is determined by (6.5). Applying (4.4) to the slow shock wave and using $\psi_1' \leq s$, one has

$$\psi_2'/\psi_1' = (1/\eta') + 0(s'\psi_1'/\eta') \quad (6.7)$$

where $0(s'\psi_1'/\eta') = 0(s^2)$. Hence, from (4.14, 6.2, and 6.7),

$$\alpha_w/s = 1 - (1/\eta) + (\psi_1'/s)[1 - (1/\eta')] \quad (6.8)$$

From (6.8, 6.4, and 2.27),

$$\alpha_w/s = (1 - xA)^{-1}[(1 - x)A + (1 - A)^{1/2}(1 - x')A'/(1 - x'A')^{1/2}] \quad (6.9)$$

From (2.21, 4.1, and 6.9),

$$M_1\alpha_w = [(2 \cdot 3^*/5^*)Q(1 - A)/(1 - xA)]^{1/2} \times [(1 - x)A(1 - A)^{-1/2} + (1 - x')A'(1 - x'A')^{-1/2}] \quad (6.10)$$

The condition $\phi > 0$ stated in Sec. 4.5 is expressed by using (6.5) as

$$A < x'A' < 1 \quad (6.11)$$

or

$$A > x'A' > 1 \quad (6.12)$$

These are important conditions to determine whether or not the solutions are proper.

In the case of OHD-OHD or MHD-MHD type, let us estimate the right-hand side of Eq. (6.10) using (3.9) or (3.10). Assuming the conditions (3.7, 3.8, and 3.8a) and the corresponding ones to the slow waves, one finds that the estimation gives $0(s^0)$. The left-hand side of (6.10) is $0(s^2)$, so that there is no solution in the case of $s^2 \ll 1$. Hence, only the OHD-MHD and MHD-OHD types are examined.

2. OHD-MHD Type

In this case, A and A' at the fast and slow shock waves take the values given by (3.9) and (3.10):

$$A = A_+ = -(5^* - 2\cdot 3^* Q)/[2(4^* - x)Q] \quad (6.13)$$

$$A' = A_-' = [3^*(x'+1)Q'/(5^* - 2\cdot 3^* Q')]s'^2 \quad (6.14)$$

One will find the conditions corresponding to (3.7, 3.8, and 3.8a) at the slow shock wave. From (6.1), since $Q' = Q_2$,

$$5^* - 2\cdot 3^* Q' \doteq (5^* + 5^* X_+ - 2\cdot 3^* Q)/(1 + X_+) \quad (6.15)$$

Hence, the condition corresponding to (3.7) is given by

$$Q_c + (1/3^*)(Q_c - Q_b) - Q = 0(s^0) \quad (6.16)$$

One gets $s' = 0(s)$ from (6.3) so that the conditions corresponding to (3 8) and (3.8a) are given by

$$|x'| \leq 0(s^0) \quad (6.17)$$

$$4^* - x' = 0(s^0) \quad (6.17a)$$

One assumes all of these six conditions.

Then one can use the formulas (3.17) and (3.18) to express η_-' and X_-'. From (6.10), one has $A_-' = 0(s^2)$ so that, using the condition (6.17), one has

$$M_1\alpha_w = (2\cdot 3^* Q/5^*)^{1/2}(1 - x)A_+/(1 - xA_+)^{1/2} \quad (6.18)$$

This is transformed to

$$M_1\alpha_w = (2\cdot 3^*/5^*)^{1/2}(Q_{c+} - Q_{b+})/Q_{c+}^{1/2} \quad (6.19)$$

using (3.14) and (3.15). Using (3.13), one finds

$$X_+ = X_0 = (2\cdot 5^*/4^*)(4^* M_1\alpha_w/2\cdot 3^*)^2\{1 + [1 + (2\cdot 3^*/4^* M_1\alpha_w)^2]^{1/2}\} \quad (6.20)$$

where the suffix 0 means the value when the magnetic field is not applied. It is seen that X_+ depends on $M_1\alpha_w$ only. From (6.20) and (3.13),

$$Q_{b+} = Q_b = (1/2\cdot 3^*)(5^* + X_0)$$
$$Q_{c+} = Q_c = (1/2\cdot 3^*)(5^* + 4^* X_0) \quad (6.21)$$

where Q_b and Q_c mean the values $(Q_{b+})_0$ and $(Q_{c+})_0$.

Now one determines the region of Q in which the OHD-MHD type of solution exists. Since $x'A_-' = 0(s^2)$, (6.12) is not satisfied. From (6.11), one has $A_+ < 0(s^2)$. Using (6.13, 3.7, and 3.8a) one has $A_+ = 0(s^0)$. From these two, one concludes that $A_+ < 0$. Hence, using (2.27) and $\eta > 0$, one has $x > 1$. Referring to (3.12), this condition is satisfied only when

$$Q < Q_b \quad (6.22)$$

One now determines the physical quantities in the region $Q < Q_b$. From (6.8, 3.17, and 3.16),

$$\alpha\eta/s = 1 - 1/\eta_+ = 1 - Q_b/Q_c \quad (6.23)$$

From (4.1, 2.21, and 3.15),

$$M_1 s = (2\cdot 3^* Q_c/5^*)^{1/2} \quad (6.24)$$

From (6.23) and (6.24),

$$s = s_0 = \alpha_w Q_c/(Q_c - Q_b) = M_1^{-1}(2\cdot 3^* Q_c/5^*)^{1/2} \quad (6.25)$$

Referring to (6.25) and (6.20), it is seen that the position s and the strength $(1 + X_+)$ of the fast shock wave are independent of Q. On the other hand, from (3.12),

$$x = (Q_c - Q)/(Q_b - Q) \quad (6.26)$$

so that the direction and, hence, the strength of the magnetic field at the fast shock wave depend on the strength of the magnetic field. The flow direction behind the shock depends on Q as (4.4) shows, so that this is a remarkable effect of the magnetic field on the OHD shock wave.

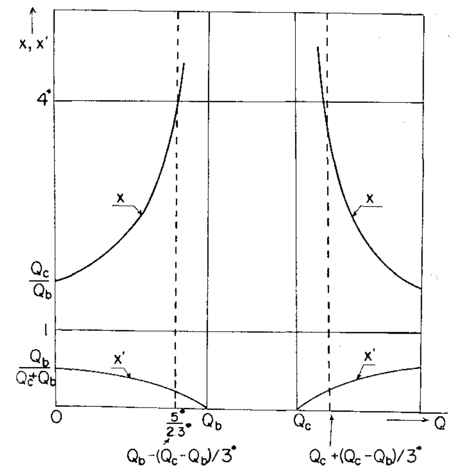

Fig. 4 x **vs** Q **and** x' **vs** Q **for definite** $M_1\alpha_w$**.**

The angle Ψ' of slow shock wave to the uniform flow is found, by using (4.16, 4.15, 6.5, 3.16, 3.14), and $x'A_-' = 0(s^2)$, to be

$$\Psi'/s_0 = 1 - (Q_b/Q_c)[1 - (Q/Q_b)^{1/2}] \quad (6.27)$$

This is also expressed by

$$\Psi'/s_0 = (\alpha_w/s_0) + (1 - \alpha_w/s_0)(Q/Q_b)^{1/2} \quad (6.28)$$

From (6.4), (2.27), $x'A' = 0(s^2)$, (3.14), and (3.15),

$$\psi_1'/s = (1 - A)^{1/2}/(1 - xA) = (Q_bQ)^{1/2}/Q_c \quad (6.29)$$

From (6.7) and (3.17),

$$\psi_2'/\psi_1' = 1 + 0(\psi_1's, s^2Q) \quad (6.30)$$

From (6.29) one has $\psi_1' = 0(sQ^{1/2})$, so that $\psi_2' = 0(sQ^{1/2})$ from (6.30) and also $\psi_1' - \psi_2' = 0(s^3Q)$. These estimations can be easily derived from (5.4) and (5.5) if one assumes $s^2 \ll 1$ and $x = 0(s^0)$ in these equations.

From (6.6, 3.12, 3.16, and 6.29),

$$x' = [Q_c - Q_b + (Q_bQ)^{1/2}]/[Q_c(Q_c - Q)(Q_b - Q)^{-1} - Q_b + (Q_bQ)^{1/2}] \quad (6.31)$$

From (6.3) and (4.16),

$$s'/s_0 = x - 1 + \Psi'/s_0 \quad (6.32)$$

One then uses (6.26) and (6.31) to obtain x and x', which are shown in Fig. 4 as the function of Q for given $M_1\alpha_w$ or X_0. The solutions of the OHD-MHD type correspond to the region $Q < Q_b$ by (6.22), where Q_b is determined by (3.13) and (6.20). Figure 5 ($Q < Q_b$) shows s/s_0, Ψ'/s_0, and s'/s_0 determined by (6.25, 6.28, 6.32, and 6.26). Figures 4 and 5 are explained again in Sec. 6.3. Since $A_-' = 0(s^2)$ under the condition (6.16), by using the same argument given in Sec. 5,

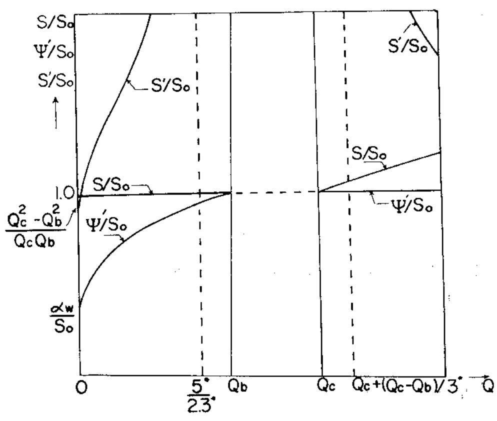

Fig. 5 s/s_0**,** Ψ'/s_0**, and** s'/s_0 **vs** Q**.**

the slow shock wave has the same propagation speed as the Alfvén wave in the direction of $\mathbf{B}_1'$ and leaves from the apex of the wedge in the flow $\mathbf{U}_1'$.

3. MHD-OHD Type

In this case, the fast shock wave is the MHD shock wave so that, from (3.10),

$$A = A_- = 0(s^2) \tag{6.33}$$

when the condition (3.7) is satisfied. Hence, from (6.10),

$$M_1\alpha_w = (2\cdot 3^*/5^*Q)^{1/2} = (1 - x')A_+'/(1 - x'A_+')^{1/2} \tag{6.34}$$

This has the same form as (6.18). Since (3.11–3.13) are satisfied by the slow shock wave in the same form, (6.34) is transformed to the same form as (6.19). Hence, from (6.20) and (6.21),

$$X_+' = X_0 \qquad Q_{b+}' = Q_b \qquad Q_{c+}' = Q_c \tag{6.35}$$

Next, the region in which the MHD-OHD type of solution exists is determined. (6.12) is not satisfied by (6.33). Hence, from (6.11), following the rule to express the inequality omitting $0(s^2)$, one has

$$1 > A_+'x' > 0 \tag{6.36}$$

Here x' is found from (3.12) and (6.35) to be

$$x' = (Q - Q_c)/(Q - Q_b) \tag{6.37}$$

Using (3.9, 6.37, and 3.13), one gets

$$A_+' = (Q - Q_b)/Q$$

where the condition (3.7) that $5^* - 2\cdot 3^*Q$ is not $o(s^0)$ was used. From (6.17) and (6.37), one sees that $Q - Q_b$ is not $o(s^0)$, so that from the foregoing three equations one gets $1 > (Q - Q_c)/Q > 0$ or

$$Q > Q_c \tag{6.38}$$

The physical quantities in the region $Q > Q_c$ are determined. From (2.21, 3.8, and 6.33), one has $m_1^2 = 0(s^0)$. Hence, from (4.1),

$$M_1 s = (2\cdot 3^*Q/5^*)^{1/2} \tag{6.39}$$

From (6.39), using (2.10), (4.15), and $M_1 = U_1/c_1$, one has

$$\alpha_1/U_1 = s = \Psi \tag{6.40}$$

This means that the position of the fast shock wave is determined by the Alfvén wave of speed α_1 in the uniform flow of speed $\mathbf{U}_1$. It is independent of the OHD shock wave and the half angle α_w of the apex of the wedge. It is interesting to note that, according to the Friedrichs diagram, the phase velocity of the Alfvén wave has an inhomogeneity with respect to the direction of the magnetic field, but if $U_1 \gg \alpha_1$ or $\alpha_1 \gg c_1$, the angle between the Alfvén shock wave and the uniform flow is given approximately by (6.40).[6]

Using (6.39) and (6.25), one has

$$s/s_0 = (Q/Q_c)^{1/2} \tag{6.41}$$

From (6.3) and (6.6),

$$s'/s = x - \phi/s = (1 - \phi/s)/x' \tag{6.42}$$

From (4.4, 4.2, and 3.17), one gets $\psi_2/s = 1$. From this equation and (4.7, 6.4, 3.17, 3.15, and 6.33), one has

$$\psi_1'/s = 1 - \phi/s = (Q_c/Q)^{1/2} \tag{6.43}$$

From (4.16, 6.41, and 6.43),

$$\Psi'/s_0 = (s - \phi)/s_0 = 1 \tag{6.44}$$

One sees, referring to (6.44) and (6.35), that the position and the strength of the slow shock wave do not depend on Q.

Using (6.41–6.43), one has

$$s'/s_0 = 1/x' \tag{6.45}$$

From (6.42, 6.43, and 6.37), one has

$$x = 1 + (Q_c/Q)^{1/2}(Q_c - Q_b)/(Q - Q_c) \tag{6.46}$$

One sees, referring to (6.46, 6.21, and 6.20), that the kink of the magnetic field line at the fast MHD shock wave is determined by Q and $M_1\alpha_w$.

Using (6.46, 6.37, and 6.38), one gets Fig. 4 ($Q > Q_c$) as x and x' vs Q for given $M_1\alpha_w$. Using (6.41, 6.44, 6.45, and 6.37), one finds s/s_0, Ψ'/s_0, and s'/s_0 as functions of Q; they are shown in Fig. 5 ($Q > Q_c$).

The results in the whole region of Q are summarized in the lowest order of s^2 (see Figs. 4 and 5). By assuming $s^2 \ll 1$, the normal components of the velocity, v_{1x}, v_{1x}', etc., are of the order of sU_1, but the pressure ratio $(1 + X_0)$ can be chosen arbitrarily. The parallel components of the magnetic field, B_{1y}, B_{1y}', etc., are of the order of sB_1. Insofar as Q satisfies (6.34), it may have a large value. For any value of Q, one sees that $x > 1$, so that the magnetic field line is bent toward the apex of the wedge inside the fast shock wave, and, referring to (2.19), the magnetic field behind the fast shock wave is stronger than that in front of it. In the MHD-OHD type, just as in the OHD-MHD type, the position and the strength of the OHD shock wave are independent of Q.

However, a remarkable effect of the magnetic field appears in the change of x or the ratio of the parallel components of the magnetic field, since $x = B_{2y}/B_{1y}$. The deflection of the direction of the magnetic field, $\theta_2 - \theta_1$, is of the order of $(x - 1)s$ because

$$\tan(\theta_2 - \theta_1) = (x - 1)/(s^{-1} + s) \approx (x - 1)s$$

Hence, the ratio $(\theta_2 - \theta_1)/\theta_1$ depends remarkably on the strength of the magnetic field. These effects of the magnetic field on the OHD shock wave do not depend on the MHD shock wave. Concerning the MHD shock wave, note that it arises at the surface of the wedge when $Q \to 0$, and it moves away from the surface at an angle proportional to $sQ^{1/2}$ [(6.29) and (6.30)] and coincides with the OHD shock wave when $Q = Q_b$. The position of the MHD shock wave is affected strongly by the wedge and the OHD shock wave, since it is made up of the Alfvén wave, which leaves the apex of the wedge in the flow of velocity $\mathbf{U}_1'$ in the direction of $\mathbf{B}_1'$ (see the discussion at the end of Sec. 6.2.). In the region $Q > Q_c$, the MHD shock wave appears preceding the OHD shock wave with the propagation speed the same as the Alfvén speed [see (6.40)], so that its position is independent of the OHD shock wave. However, the direction of the magnetic field depends on the OHD shock wave. At $Q = Q_c$, the two shock waves coalesce.

There is no solution in the region $Q_b < Q < Q_c$. There the MHD shock wave may be detached from the apex of the wedge. Cabannes[7] has found the same phenomenon in the aligned-magnetic-field case in the uniform flow past a wedge.

In the case $Q = Q_b$ or $Q = Q_c$, one has $x \to \infty$. This may be allowable if $s^2 \to 0$. Hence, there is the possibility that the strength of the magnetic field increases infinitely, but this is not so. Namely, from the condition that (2.25) have real roots under the conditions $s^2 \ll 1$ and $x \gg 1$, one has

$$8\cdot 3^*x^2s^2Q^2 < (2\cdot 3^* - 5^* - xs^2Q)^2 < (2\cdot 3^*Q)^2$$

so that

$$x^2s^2 < 3^*/2 \tag{6.47}$$

From (2.19), one has

$$T^2 < 1 + (xs)^2 < 5^*/2$$

so that, near the value of $Q = Q_b$ or $Q = Q_c$,

$$B_2/B_1 < (5^*/2)^{1/2} \tag{6.48}$$

The foregoing solutions do not hold in the region where the conditions (3.7, 3.8, 3.8a, 6.16, 6.17, and 6.17a) are not satisfied. A few remarks are added to these conditions. The region where the condition (3.7) does not hold is contained in the region where the OHD-MHD type of solution exists, since it exists in the region $Q < Q_b$, and $5^* - 2\cdot 3^*Q = 0$ means that

$$Q = 5^*/2\cdot 3^* = Q_b - (Q_c - Q_b)/3^*$$

When $Q = 5^*/2\cdot 3^*$, one has

$$\alpha_1 = c_1 \tag{6.49}$$

by (2.10). Namely, when the Alfvén velocity and the sound velocity are the same in the undisturbed flow, the solution may be singular. However, there is no singularity when the Alfvén velocity is equal to the freestream velocity. The condition (3.8) concerns the solution when two shock waves coalesce. When $x = 4^*$, one has $Q = 5^*/2\cdot 3^*$ from (6.26), so that (3.8a) is involved in (3.7) in the region of the MHD-OHD type of solution. This condition is not equal to (3.8a) in $Q > Q_c$. Conditions (6.17) and (6.17a) are always satisfied, which is easily verified in Fig. 4. The regions avoided by the conditions (3.7, 3.8, 3.8a, and 6.16) in the case of small-angle shock waves and the case of general angles of shock waves await further investigation.

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A Solution for the Nonequilibrium Flat-Plate Boundary Layer

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An analytic solution is presented for the boundary-layer flow of a dissociating diatomic gas over a flat plate. Chemical reaction is assumed to be produced solely by viscous heating within the boundary layer. Near the leading edge, where the gas is far from equilibrium, the dominant chemical process is dissociation, whereas the inverse process of recombination is negligible. The dissociation reaction takes place within a rather narrow region centered on the maximum of the frozen temperature profile. By taking advantage of the exponential temperature dependence of the dissociation rate, simple expressions for the temperature and concentration are derived. In particular, the initial rate of accumulation of atoms at a noncatalytic surface is given by the reaction rate at the maximum frozen temperature, modified to account for convection and diffusion. The physical interpretation of these results suggests a simple formula for making predictions in cases where more complicated chemistry occurs. This formula accurately reproduces the results of numerical solutions.

Nomenclature

A_1 = $h_1^{(0)}/u_\infty^2$
c = atom mass fraction
C_P = specific heat at constant pressure
D_{12} = binary diffusion coefficient
$f(\eta)$ = Blasius function
F = function defined by Eq. (36)
$\mathfrak{F}$ = function defined by Eq. (42)
g,G = functions defined by Eq. (29)
h,H = functions defined by Eq. (29)
h_i = perfect-gas enthalpy of ith species
$h_i^{(0)}$ = heat of formation of ith species
k = thermal conductivity
k_d = dissociation rate constant
k_r = recombination rate constant
K = $4k_{r0}/R^2$
l = $\rho\mu/(\rho\mu)_w$
Le = Lewis number, $D_{12}\rho C_P/k$
$m_{1,2}$ = atomic, molecular weight
p = pressure
$P_k(\eta)$ = functions defined by Eq. (18)
Pr = Prandtl number, $\mu C_P/k$
q = heat-transfer rate
Q = dimensionless heat-transfer coefficient, Eq. (14)
r_D = forward reaction rate, $4k_r\rho(p/RT)^2 \times [c_{eq}^2/(1 - c_{eq}^2)](1 - c)$
r_R = reverse reaction rate, $4k_r\rho(p/RT)^2[c^2/(1 + c)]$
R = universal gas constant
S = temperature dependence of recombination rate constant, $k_r = k_{r0}T^{-S}$
$s(\eta)$ = function defined by Eq. (26)
T = temperature
T_* = reference temperature, u_∞^2/C_P
T_D = characteristic dissociation temperature, $2h_1^0 m_1/R$
u = x component of velocity

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